

UNIVERSITY OF MISSOURI

AN EXTENSION OF MASCHKE'S SYMBOLISM

NOLA LEE ANDERSON

A THESIS IN MATHEMATICS
PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

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An Extension of Maschke's Symbolism.*

BY NOLA LEE ANDERSON.

1. *Introduction.* This paper considers a generalization of the symbolic method of Maschke for representing the invariants of a quadratic differential form $\sum g_{ij}dx^i dx^j$. According to this method, the coefficients g_{ij} are represented symbolically as products of $f_i f_j$.

Maschke regarded f_i and f_j as partial derivatives, $\partial f/\partial x^i$, $\partial f/\partial x^j$, of a symbolic function f , and placed

$$(1) \quad A = \sum g_{ij} dx^i dx^j = (df)^2 = (\sum f_i dx^i)^2.$$

The possibility of a similar symbolic representation in which it is not assumed that $\partial f_i/\partial x^j = \partial f_j/\partial x^i$ has been suggested by Wilson and Moore.† This is the generalization that is to be considered in this paper.

As might be expected this generalization does not affect the mode of representing first order differential parameters. It is found, however, that invariant forms of higher order can best be regarded as invariants of the set of quantities g_{ij} together with certain other sets.

The two dimensional case is discussed in some detail. A geometric interpretation is introduced,‡ and certain invariant vectors related to a surface are studied.

The $f_1(u, v)$ and $f_2(u, v)$ are here regarded as any pair of independent vectors, satisfying the relation (1), associated with the point u, v of the surface. We find that the plane (called the base plane) which they determine has many properties analogous to properties of the tangent plane in the ordinary case. The effect of varying the orientation of the vectors f_1 and f_2 in the base plane is considered only in a special case.

Normals to this plane which are expressible in terms of the first deriva-

* Presented to the American Mathematical Society, November 27, 1926.

† "Differential Geometry of Two Dimensional Surface in Hyperspace," *Proceedings of the American Academy of Arts and Sciences*, Vol. 52 (1916), p. 294.

‡ A geometric interpretation of the Symbolic Method of Maschke was made and studied in detail by L. Ingold in the paper "A Symbolic Treatment of the Geometry of Hyperspace," *Transactions of the American Mathematical Society*, Vol. 27 (1925), pp. 574-599.

tives of f_1 and f_2 are called the first normals relative to the base plane. In general, there are four independent first normals to the base plane instead of three as in the usual case.*

The analogues of the various curvature vectors associated with the curves $a(u, v) = \text{constant}$ are studied. These vectors in the main have properties very similar to the properties of the usual curvature vectors. There are, however, certain novel features. In particular we mention the appearance of a new invariant vector.

2. *First order invariants.* In order to represent expressions of higher than the first degree in g_{ij} by means of symbols, Maschke introduced symbols ϕ_i, ψ_i , etc. equivalent to the f_i . Thus, $g_{ij}g_{rs}$ might be written symbolically as $f_i f_j \cdot \phi_r \phi_s$, and so on.

The fundamental theorem states that if $A^1, \dots, A^n$ are any n invariants of the quadratic differential form $G \equiv \sum g_{ij} dx^i dx^j$ in n variables, then β times their Jacobian is also an invariant where β is the reciprocal of the square root of the discriminant of G . This is written in the abbreviated form $(A^1, A^2, \dots, A^n)$, where the factor β is understood.

In the usual sense, arbitrary functions are invariants so that some or all of the A 's may stand for arbitrary functions of $x^1, \dots, x^n$. This applies to the symbolic functions f , and, likewise, some or all of the A 's may stand for any of the equivalent symbolic functions f, ϕ, ψ , etc. Again, the product of any number of the invariants is invariant. This fact enables us to construct invariants involving the g_{ij} by writing products in which the symbol f occurs in two distinct factors; therefore $(f A^1 \dots A^{n-1})(f B^1 \dots B^{n-1})$ is an invariant containing the quantities g_{ij} .

In generalizing these ideas, we shall continue to use the symbolism of Maschke and let $f_i f_j = g_{ij}$, where f_i and f_j , however, are not regarded as partial derivatives. It becomes necessary, then, to give their law of transformation. It has been noticed that the argument by which it is shown that $(A^1, \dots, A^n)$ is an invariant is based chiefly on the fact that the derivatives $\partial A^i / \partial x^j$ transform according to the law

$$\partial \bar{A}^i / \partial \bar{x}^s = \sum (\partial A^i / \partial x^j) (\partial x^j / \partial \bar{x}^s)$$

when the variables are changed from $x^1, \dots, x^n$ to $\bar{x}^1, \dots, \bar{x}^n$. In accordance with this, we assume that symbols f_i follow the same law of transformation

$$(2) \quad \bar{f}_r = \sum f_j (\partial x^j / \partial \bar{x}^r)$$

with similar equations for equivalent symbols.

* See Wilson and Moore, *loc. cit.*, p. 304.

It is evident that the invariant expressions used by Maschke are still formally invariant with this extended conception of the f_i , ϕ_i , etc. Throughout this discussion, it will be understood that parenthesis expressions containing the symbolic functions have the same meaning as in Maschke's theory except that f , ϕ , etc. represent columns $f_1, f_2, \dots, f_n, \phi_1, \phi_2, \dots, \phi_n$, etc. instead of derivatives. Some of the elements of the parenthesis expressions may be parenthesis expressions themselves, as $((fa)a)(fb)$. In such cases, $(fa)_1$, means $\partial(fa)/\partial x^1$, etc., and, in general, unless the contrary is specified, the elements of a column of any parenthesis will be understood to be the partial derivatives of the quantity at the head of the column excepting only the case of f or its equivalent.

It is seen immediately that any symbolic invariant expression represents exactly the same invariant whether the f_i are regarded as derivatives or not, *provided no derivatives of the f_i, ϕ_i , etc. appear.*

3. *Triple index symbols.* The invariant expressions involving first derivatives of the fundamental quantities or of arbitrary functions are called invariants of the first order. It is clear that the generalization proposed in this paper does not affect the representation of such invariants. But other invariant expressions may occur which contain derivatives of the f_i , that is, such quantities as f_{ij} , where f_{ij} denotes the partial derivative, $\partial f^i/\partial x^j$.^{*} Then, invariants of this type are called invariants of the second order.

From the equations $f_{ij} = g_{ij}$, we find by differentiation with respect to x^i that

$$(3) \quad f_{ij}f_j = (1/2)\partial g_{jj}/\partial x^i.$$

The products $f_{ij}f_{jk}$ will be denoted by $[jk, i]$; these are analogous to the usual Christoffel triple index symbols.

If the equation $f_{ij} = g_{ij}$, where $i \neq j$, is differentiated with respect to x^k , we obtain

$$(4) \quad f_{ij}f_{jk} + f_jf_{ik} = \partial g_{ij}/\partial x^k.$$

In the usual treatment the expressions $[ij, k]$ and $[ji, k]$ are the same so that the equations (3) and (4) are sufficient to determine all of the triple index symbols in terms of the derivatives of the fundamental quantities.

Here, however, this is no longer the case. But if notations, P , Q , etc., are introduced for a sufficient number of these symbols, all may be expressed in terms of the g_{ij} and these additional quantities,

* Whenever double subscripts occur with a symbolic quantity, the second one always denotes differentiation.

$$P, Q, \dots$$

Furthermore, it is possible to express several combinations of $f_{ij}f_{km}$ and $f_i f_{jkm}$ in terms of the quantities g_{ij} together with the additional quantities just mentioned.

4. *Geometric interpretation.* A geometric interpretation of this symbolism for the two dimensional case will now be considered. Throughout the remainder of this paper, we shall let u and v denote the curvilinear coordinates of a two dimensional surface in space.* We shall also from this point on consider

$$g_{11} = E, \quad g_{12} = F, \quad g_{22} = G,$$

except in cases where it is advantageous to use the coefficients g_{ij} in the formulas.

As has been referred to previously, Maschke used the symbol f_i as $\partial f / \partial x^i$. To interpret this geometrically, $f(u, v)$ may be regarded as a vector from some arbitrary origin to points of a two dimensional surface in space. Hence, at each point, f_1 and f_2 will be considered as a pair of vectors which are the partial derivatives of f , with respect to u and v , and therefore are tangent to the parameter curves, $v = \text{constant}$ and $u = \text{constant}$, respectively. Thus, f_1 and f_2 lie in the tangent plane to the surface. In this case, of course, $f_{12} = f_{21}$.

Under the proposed generalization, the $f_1(u, v)$ and $f_2(u, v)$ are interpreted as any two vectors associated with the point u, v of the surface subject to the conditions that $f_1^2 = E$, $f_1 f_2 = F$, and $f_2^2 = G$. The products $f_i f_j$ will be regarded as scalar products. The plane which these vectors determine will be called the base plane † of the corresponding point of the surface, and the vectors themselves will occasionally be referred to as the fundamental vectors or base vectors. The base plane and the fundamental vectors $\mathbf{f}_1, \mathbf{f}_2$ ‡ will be called the basis $[\mathbf{f}_1, \mathbf{f}_2]$.

* This space is of unspecified number of dimensions.

† The introduction of an arbitrary base plane at each point of a surface in the place of the tangent plane is very similar to the introduction of an arbitrary line at each point of a curve instead of the tangent line. The effect of replacing the tangent vector to a curve by an arbitrary vector function of the arc length or other parameter was considered by W. Fr. Meyer in the paper, "Ausdehnung der Frenetschen Formeln und Verwandter auf den R_n ," *Jahresbericht der Deutschen Mathematiker Vereinigung*, Vol. 19 (1910), pp. 160-169.

A similar method has been recently employed in the study of Riemannian geometry. See Eisenhart, *Riemannian Geometry*, p. 73 and p. 107 where further references will be found.

‡ From this point on bold-faced type will be used to indicate vectors.

It should be mentioned that the formulas of the theory will depend not only on the base plane but on the orientation of the fundamental vectors as well. The study of properties independent of this orientation is not included.

Since in the general case the restriction that $\mathbf{f}_{12} = \mathbf{f}_{21}$ is removed, the quantity $\mathbf{f}_{12}$ is considered as independent of the quantity $\mathbf{f}_{21}$.

If in a special case the $\mathbf{f}_1, \mathbf{f}_2$ are the derivatives of a vector $\mathbf{f}$, they are regarded as tangents to the parameter curves of the surface, and the base plane becomes the tangent plane. This case will be known as the ordinary or usual case. It is clear, however, that when the base plane coincides with the tangent plane, the fundamental vectors $\mathbf{f}_1, \mathbf{f}_2$ need not coincide with the tangents to the parameter curves. This will be illustrated in the special case which is considered in the next articles.

5. *A special case.* We consider here the case in which the base plane coincides with the tangent plane. There are first developed a number of general formulas which will be valuable later for purposes of reduction.

From the scalar products,

$$(5) \quad \mathbf{f}_1 \mathbf{f}_1 = E, \quad \mathbf{f}_1 \mathbf{f}_2 = F, \quad \mathbf{f}_2 \mathbf{f}_2 = G,$$

we deduce by differentiation the following equations for the triple index symbols:

$$(6) \quad [11, 1] = (1/2)E_1, \quad [21, 2] = (1/2)G_1, \quad [21, 1] + [11, 2] = F_1, \\ [12, 1] = (1/2)E_2, \quad [22, 2] = (1/2)G_2, \quad [12, 2] + [22, 1] = F_2.$$

It is convenient now to introduce the additional quantities, P, Q , etc. mentioned in section 3. Therefore, we shall let $[11, 2] = P$ and $[22, 1] = Q$.

The Christoffel triple index symbols of the second kind will be used throughout this paper and are defined in the usual manner by the equations

$$(7) \quad \left\{ \begin{matrix} k \\ ij \end{matrix} \right\} = \sum g^{km} [ij, m].$$

That is,

$$\left\{ \begin{matrix} 1 \\ 12 \end{matrix} \right\} = g^{11}[12, 1] + g^{12}[12, 2], \text{ etc., where}$$

$$(8) \quad g^{11} = G\beta^2, \quad g^{12} = -F\beta^2, \quad g^{21} = -F\beta^2, \quad g^{22} = E\beta^2.$$

We may, also, express the triple index symbols of the first kind in terms of the triple index symbols of the second kind,

$$(9) \quad [ij, k] = \sum g_{km} \left\{ \begin{matrix} m \\ ij \end{matrix} \right\},$$

and by this relation we have equations of the type

$$[11, 1] = g_{11} \begin{Bmatrix} 1 \\ 11 \end{Bmatrix} + g_{12} \begin{Bmatrix} 2 \\ 11 \end{Bmatrix}.$$

By taking the second derivatives of $f_1^2 = E$, $f_2^2 = G$, and $f_1 f_2 = F$, several combinations of $f_{ik} f_{lm}$ and $f_i f_{klm}$ can be expressed in terms of E , F , G , P , Q , and their derivatives, such as

$$f_{21} f_{11} + f_2 f_{111} = P_1, \quad f_{11} f_{22} - f_{12} f_{21} = P_2 + Q_1 - F_{12}, \text{ etc.}$$

Now if the base plane coincides with the tangent plane the vectors f_1 and f_2 must be tangent vectors; they are therefore linearly expressible in terms of the vectors tangent to the parameter curves.

In the remaining portion of this section and in section 7 the notations φ_1 and φ_2 will be used to denote vectors tangent to the parameter curves $v = \text{constant}$ and $u = \text{constant}$, respectively, subject to the conditions

$$\varphi_1^2 = E, \quad \varphi_1 \varphi_2 = F, \quad \varphi_2^2 = G.$$

It is clear from these equations and equations (5) that the cosine of the angle between the vectors φ_1 and φ_2 is the same as the cosine of the angle between f_1 and f_2 , and the angles themselves are either equal or supplementary. We consider the case in which they are equal.

Let $\theta(u, v)$ denote the angle between f_1 and φ_1 which is also the angle between f_2 and φ_2 . We then have the formulas

$$f_1 \varphi_1 = E \cos \theta, \quad f_2 \varphi_2 = G \cos \theta,$$

and therefore

$$(f_1 \varphi_1)/E = (f_2 \varphi_2)/G = \cos \theta.$$

These equations enable us to determine the coefficients p , q , m , n in the following equations

$$(10) \quad f_1 = p \varphi_1 + q \varphi_2,$$

$$(11) \quad f_2 = m \varphi_1 + n \varphi_2.$$

If (10) be multiplied by φ_1 , and (11) by φ_2 , the equations

$$p = \cos \theta - q(F/E)$$

and $n = \cos \theta - m(F/G)$

are obtained.

If (10) be squared, we find after substituting the value of p , that $q = (E \sin \theta)/(EG - F^2)^{1/2}$; * consequently,

$$p = \cos \theta - (F \sin \theta)/(EG - F^2)^{1/2}.$$

* The use of the opposite sign for q is equivalent to reversing the angle θ .

Again, when (10) is multiplied by (11) and the substitutions for p , q , and m are made, we have after simplifying, $m = (-G \sin \theta)/(EG - F^2)^{1/2}$, and consequently $n = \cos \theta + (F \sin \theta)/(EG - F^2)^{1/2}$.

Equations (10) and (11) may now be written

$$\begin{aligned} f_1 &= (\cos \theta - \beta F \sin \theta) \varphi_1 + (\beta E \sin \theta) \varphi_2, \\ f_2 &= -(\beta G \sin \theta) \varphi_1 + (\cos \theta + \beta F \sin \theta) \varphi_2 \\ &\quad \text{where } \beta = 1/(EG - F^2)^{1/2}. \end{aligned}$$

From these we can compute the derivatives of the base vectors, f_1 and f_2 , with respect to u and v ,

$$\begin{aligned} f_{11} &= [\cos \theta - \beta F \sin \theta] \varphi_{11} + [\beta E \sin \theta] \varphi_{21} \\ &\quad + [\cos \theta - \beta F \sin \theta]_1 \varphi_1 + [\beta E \sin \theta]_1 \varphi_2; \\ f_{12} &= [\cos \theta - \beta F \sin \theta] \varphi_{12} + [\beta E \sin \theta] \varphi_{22} \\ &\quad + [\cos \theta - \beta F \sin \theta]_2 \varphi_1 + [\beta E \sin \theta]_2 \varphi_2; \\ f_{21} &= -[\beta G \sin \theta] \varphi_{11} + [\cos \theta + \beta F \sin \theta] \varphi_{21} \\ &\quad - [\beta G \sin \theta]_1 \varphi_1 + [\cos \theta + \beta F \sin \theta]_1 \varphi_2; \\ f_{22} &= -[\beta G \sin \theta] \varphi_{12} + [\cos \theta + \beta F \sin \theta] \varphi_{22} \\ &\quad - [\beta G \sin \theta]_2 \varphi_1 + [\cos \theta + \beta F \sin \theta]_2 \varphi_2. \end{aligned}$$

If the scalar product of f_2 and f_{11} is computed, $P = [11, 2]$ is obtained in terms of E , F , G , and θ . In a similar manner, $Q = [22, 1]$ can be found, and thus by equations (6) all of the expressions $[ij, k]$ are known.

6. *Normal vectors.* By normal vectors we mean vectors that are orthogonal to the fundamental system of vectors, that is, in this case vectors orthogonal to the base plane. The vectors f_{ij} are in general oblique to the base plane, but there exist normal vectors expressible in terms of the fundamental vectors f_1 , f_2 , and the vectors f_{11} , f_{12} , f_{21} , f_{22} . Such vectors will be called first normals. Since f_{11} , f_{12} , f_{21} , f_{22} are as a rule independent, there are in general four independent first normals. This paper is for the most part concerned only with those cases in which these are independent.

One such set of four independent normals is given by

$$(12) \quad N_{ij} = f_{ij} - \sum \left\{ \begin{matrix} m \\ ij \end{matrix} \right\} f_m.$$

It is shown easily that N_{ji} and f_k are orthogonal, i. e. $N_{ij} f_k = 0$. If (12) is multiplied by f_k ,

$$f_k N_{ij} = [ij, k] - \sum g_{km} \left\{ \begin{matrix} m \\ ij \end{matrix} \right\}.$$

Hence, it follows immediately from (9) that

$$\mathbf{f}_k \mathbf{N}_{ij} = 0.$$

There are, of course, other sets of independent first normals. In some cases the following set is more convenient,

$$(13) \quad \mathbf{a} = \phi_1(\mathbf{f}\phi)_1, \quad \mathbf{b} = \phi_2(\mathbf{f}\phi)_2, \quad \mathbf{c} = \phi_1(\mathbf{f}\phi)_2, \quad \mathbf{d} = \phi_2(\mathbf{f}\phi)_1.*$$

It is readily proved, for example, that $\phi_1(\mathbf{f}\phi)_1$ is a normal vector.

$$(\mathbf{f}\phi)_1 = \beta_1[\mathbf{f}_1\phi_2 - \mathbf{f}_2\phi_1] + \beta[\mathbf{f}_{11}\phi_2 - \mathbf{f}_{21}\phi_1 + \mathbf{f}_1\phi_{21} - \mathbf{f}_2\phi_{11}],$$

and if this be multiplied by ϕ_1 ,

$$\phi_1(\mathbf{f}\phi)_1 = \beta_1[F\mathbf{f}_1 - E\mathbf{f}_2] + \beta[F\mathbf{f}_{11} - E\mathbf{f}_{21} + [21, 1]\mathbf{f}_1 - [11, 1]\mathbf{f}_2].$$

Simplifying and substituting the values of $\mathbf{f}_{11}$ and $\mathbf{f}_{21}$ from (12),

$$\begin{aligned} \phi_1(\mathbf{f}\phi)_1 &= \{\beta_1 F + \beta[21, 1]\}\mathbf{f}_1 - \{\beta_1 E_2 + \beta[11, 1]\}\mathbf{f}_2 \\ &+ \beta F(N_{11} + \left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} \mathbf{f}_1 + \left\{ \begin{matrix} 2 \\ 11 \end{matrix} \right\} \mathbf{f}_2) - \beta E(N_{21} + \left\{ \begin{matrix} 1 \\ 21 \end{matrix} \right\} \mathbf{f}_1 + \left\{ \begin{matrix} 2 \\ 21 \end{matrix} \right\} \mathbf{f}_2) \\ &= (\beta_1 F + \beta[21, 1] + \beta F \left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} - \beta E \left\{ \begin{matrix} 1 \\ 21 \end{matrix} \right\})\mathbf{f}_1 \\ &- (\beta_1 E + \beta[11, 1] - \beta F \left\{ \begin{matrix} 2 \\ 11 \end{matrix} \right\} + \beta E \left\{ \begin{matrix} 2 \\ 21 \end{matrix} \right\})\mathbf{f}_2 + \beta F N_{11} - \beta E N_{21}. \end{aligned}$$

Since $\beta = 1/(EG - F^2)^{1/2}$,

$$\begin{aligned} \beta_1 &= -(E_1 G + EG_1 - 2FF_1)/2(EG - F^2)^{3/2} \\ &= -\beta^3[(1/2)E_1 G + (1/2)G_1 E - FF_1], \end{aligned}$$

and by (6),

$$\beta_1 = -\beta^3 G[11, 1] - \beta^3 E[21, 2] + \beta^3 F[21, 1] + \beta^3 F[11, 2].$$

Now by substituting this value β_1 in the last equation for $\phi_1(\mathbf{f}\phi)_1$, we find

$$\mathbf{a} = \phi_1(\mathbf{f}\phi)_1 = \beta(FN_{11} - EN_{21}).$$

In like manner,

$$(14) \quad \begin{aligned} \mathbf{b} &= \phi_2(\mathbf{f}\phi)_2 = \beta(GN_{12} - FN_{22}), \\ \mathbf{c} &= \phi_1(\mathbf{f}\phi)_2 = \beta(FN_{12} - EN_{22}), \\ \mathbf{d} &= \phi_2(\mathbf{f}\phi)_1 = \beta(GN_{11} - FN_{21}). \end{aligned}$$

It is obvious that these are normal vectors.

* $(\mathbf{f}\phi)_1$, and $(\mathbf{f}\phi)_2$ denote the partial derivatives $\partial(\mathbf{f}\phi)/\partial u$ and $\partial(\mathbf{f}\phi)/\partial v$. It seems preferable in formulas of this kind to print only the notation $\mathbf{f}$, which appears just once, in bold-faced type to indicate the vectorial character of the expression.

We can now show without difficulty that the invariant vectors

$$(15) \quad (\phi a)((f\phi)a), \quad (\phi a)(\psi a)((f\phi)\psi), \quad (\psi\phi)(\psi a)((f\phi)a),$$

where a denotes a function of u and v , are normal vectors since it is possible to express them linearly in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$.

For example,

$$\begin{aligned} (\phi a)((f\phi)a) &= \beta[\phi_1 a_2 - \phi_2 a_1] \beta[(f\phi)_1 a_2 - (f\phi)_2 a_1] \\ &= \beta^2[\phi_1 (f\phi)_1 a_2^2 - \phi_1 (f\phi)_2 a_1 a_2 - \phi_2 (f\phi)_1 a_1 a_2 \\ &\quad - \phi_2 (f\phi)_2 a_1^2] \\ &= a_2^2 \mathbf{a} - a_1 a_2 (\mathbf{c} - \mathbf{d}) + a_1^2 \mathbf{b}. \end{aligned}$$

By expanding in a similar manner it is found that the invariant vector $(\psi\phi)(\psi(f\phi))$, is also a normal vector. (In the ordinary case this vector vanishes identically.)*

$$\begin{aligned} \text{Thus, } (\psi\phi)(\psi(f\phi)) &= \beta^2\{[\psi_1 \phi_2 - \psi_2 \phi_1][\psi_1 (f\phi)_2 - \psi_2 (f\phi)_1]\} \\ &= \beta^2(E\mathbf{b} - F\mathbf{d} - F\mathbf{c} + G\mathbf{a}). \end{aligned}$$

If the values of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ from equations (14) are substituted in this, we obtain

$$(16) \quad (\psi\phi)(\psi(f\phi)) = \beta[N_{12} - N_{21}] = \mathbf{j}.$$

There is another invariant normal vector,

$$\begin{aligned} ((f\phi)\phi) &= \beta[(f\phi)_1 \phi_2 - (f\phi)_2 \phi_1], \text{ or by (13),} \\ (17) \quad ((f\phi)\phi) &= \beta(\mathbf{d} - \mathbf{c}) = \mathbf{h}^*. \end{aligned}$$

It should be observed that the vectors $\mathbf{h}^*$ and $\mathbf{j}$ do not depend on an arbitrary function a .

7. *Application to special case.* It will be recalled that in the discussion of section 5 the base plane was made to coincide with the tangent plane to the surface and the fundamental vectors $\mathbf{f}_1$, $\mathbf{f}_2$ to differ from the tangent vectors $\boldsymbol{\varphi}_1$, $\boldsymbol{\varphi}_2$ only by a rotation through the angle θ . In this special case, it is possible to express the invariant normal vectors $\mathbf{j}$ and $\mathbf{h}^*$ in terms of θ and $\mathbf{h}$, where $\mathbf{h} = \beta(\mathbf{d} - \mathbf{c})$ in the ordinary case.

First, let us expand $\mathbf{h}$, regarding the $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ as tangent vectors, in the usual case.

* See L. Ingold, *loc. cit.*, p. 580.

$$\begin{aligned}
\mathbf{h} &= \beta(\mathbf{d} - \mathbf{c}) = \beta[(\varphi\psi)_1\psi_2 - (\varphi\psi)_2\psi_1] \\
&= \beta\{\beta_1[\varphi_1\psi_2 - \varphi_2\psi_1]\psi_2 - \beta_2[\varphi_1\psi_2 - \varphi_2\psi_1]\psi_1 \\
&\quad + \beta\psi_2[\varphi_{11}\psi_2 - \varphi_{21}\psi_1 + \varphi_1\psi_{21} - \varphi_2\psi_{11}] \\
&\quad - \beta\psi_1[\varphi_{12}\psi_2 - \varphi_{22}\psi_1 + \varphi_1\psi_{22} - \varphi_2\psi_{12}]\},
\end{aligned}$$

where ψ_1, ψ_2 are equivalent to φ_1, φ_2 .

Simplifying and collecting,

$$\begin{aligned}
\mathbf{h} &= \beta\beta_1(G\varphi_1 - F\varphi_2) - \beta\beta_2(F\varphi_1 - E\varphi_2) \\
&\quad + \beta^2(G\varphi_{11} - 2F\varphi_{12} + E\varphi_{22}) \\
&\quad + \beta^2\varphi_1([21, 2] + [22, 1]) - \beta^2\varphi_2([11, 2] + [12, 1]),
\end{aligned}$$

where we make use of the fact $\varphi_{12} = \varphi_{21}$.

Thus, $\mathbf{h} = \beta^2[G\varphi_{11} - 2F\varphi_{12} + E\varphi_{22}] + \text{terms in } \varphi_1 \text{ and } \varphi_2$.

Next, we shall consider the invariant normal vector $\mathbf{j} = (\psi\phi)(\psi(f\phi)) = \beta[N_{12} - N_{21}]$. When the values for N_{12} and N_{21} from (12) are substituted in this equation,

$$\mathbf{j} = \beta[f_{12} - \left\{ \begin{matrix} 1 \\ 12 \end{matrix} \right\} f_1 - \left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\} f_2 - f_{21} + \left\{ \begin{matrix} 1 \\ 21 \end{matrix} \right\} f_1 + \left\{ \begin{matrix} 2 \\ 21 \end{matrix} \right\} f_2].$$

Therefore,

$$\mathbf{j} = \beta[f_{12} - f_{21}] + \text{terms in } f_1 \text{ and } f_2.$$

From the values for f_{12} and f_{21} obtained in section 5,

$$f_{12} - f_{21} = \beta \sin \theta [G\varphi_{11} - 2F\varphi_{12} + E\varphi_{22}] + \text{terms in } \varphi_1 \text{ and } \varphi_2,$$

where φ_1 and φ_2 denote the tangent vectors to the parameter curves. It will, also, be recalled that f_1 and f_2 can be expressed in terms of φ_1 and φ_2 . As a result of this, $\mathbf{j}$ can be written

$$\mathbf{j} = (\sin \theta)\mathbf{h} + \text{terms in } f_1 \text{ and } f_2.$$

Since $\mathbf{j}$ and $\mathbf{h}$ are normal vectors, the terms involving the tangent vectors, f_1 and f_2 , will vanish. Hence,

$$(18) \quad \mathbf{j} = (\sin \theta)\mathbf{h}.$$

Lastly, we shall consider the normal vector $\mathbf{h}^* = ((f'f')f')$ with f' equivalent to f . When it is expanded in a manner similar to the expansion of $\mathbf{j}$,

$$\mathbf{h}^* = \beta^2 \cos \theta \{G\varphi_{11} - 2F\varphi_{12} + E\varphi_{22}\} + \text{terms in } f_1 \text{ and } f_2,$$

which may be written

$$\mathbf{h}^* = (\cos \theta)\mathbf{h} + \text{terms in } f_1 \text{ and } f_2.$$

By the same reasoning as in the above case,

$$(19) \quad \mathbf{h}^* = (\cos \theta) \mathbf{h}.$$

8. *Polar directions.* We shall now consider the curves $a = \text{constant}$ on the two dimensional surface whose coördinates are u and v .

If the quantity V is a scalar function of u and v , the invariant $(Va)/[(fa)^2]^{1/2}$, is the same as it was before this generalization was made. That is, $(Va)/[(fa)^2]^{1/2} = dV/ds$, which is the derivative of the scalar quantity V with respect to arc length s along the curves $a = \text{constant}$.*

But this is not true with $(fa)/[(fa)^2]^{1/2}$, which will be denoted by $\mathbf{t}$, because $\mathbf{f}_1$ and $\mathbf{f}_2$ are not $\partial \mathbf{f}/\partial u$ and $\partial \mathbf{f}/\partial v$. This, however, is an invariant vector lying in the base plane. Any scalar multiple of this vector (fa) will be called a polar vector of the curve $a = \text{constant}$ with respect to the basis. The direction of this polar vector may be called the polar direction. There is a polar direction corresponding to each point of the curve $a = \text{constant}$.

It should be noted that by this definition of polar vector, the vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ are the polar vectors of the parameter curves $v = \text{constant}$ and $u = \text{constant}$, respectively.

If this same operation is performed on $\mathbf{t}$, $(\mathbf{t}a)/[(fa)^2]^{1/2}$ results, which is really $d\mathbf{t}/ds$ since $\mathbf{t}_1$ and $\mathbf{t}_2$ are $\partial \mathbf{t}/\partial u$ and $\partial \mathbf{t}/\partial v$, respectively. This vector is orthogonal to the polar vector, because $\mathbf{t}^2 = 1$ and $\mathbf{t}(d\mathbf{t}/ds) = 0$.

The vector $(\mathbf{t}a)/[(fa)^2]^{1/2}$ will be called a principal normal to $a = \text{constant}$, relative to the basis, and its magnitude will be known as the first curvature relative to the basis. Usually these expressions will be called simply the principal normal, and the first curvature, the relation to the basis, being understood. In general, as has already been mentioned, a normal vector means a vector orthogonal to the base plane.

Instead of the polar vector (fa) to the curves $a = \text{constant}$, we consider any vector function of u and v in the base plane, say $\alpha_1 \mathbf{f}_1 + \alpha_2 \mathbf{f}_2$, where α_1 and α_2 are functions of the coördinates. Suppose there exists a function b such that $\partial b/\partial u = -\lambda \alpha_2$ and $\partial b/\partial v = \lambda \alpha_1$, where λ is a proportionality factor. In this case, $(Vb)/[(fb)^2]^{1/2} = dV/ds$, where s is the arc length of the curve $b = \text{constant}$. $(fb) = \lambda(\alpha_1 \mathbf{f}_1 + \alpha_2 \mathbf{f}_2)$ which is the polar vector to $b = \text{constant}$, and $(Vb) = \lambda(V_1 \alpha_1 + V_2 \alpha_2)$, hence,

$$(Vb)/[(fb)^2]^{1/2} = (V_1 \alpha_1 + V_2 \alpha_2)/[(f_1 \alpha_1 + f_2 \alpha_2)^2]^{1/2}.$$

If $\alpha_1 \mathbf{f}_1 + \alpha_2 \mathbf{f}_2$ is the polar vector of a curve $b = \text{constant}$, it follows that

* L. Ingold, *loc. cit.*, p. 583.

the derivatives of any point function with respect to arc length along $b = \text{constant}$ is obtained by writing V_1, V_2 (that is $\partial V/\partial u, \partial V/\partial v$) instead of f_1, f_2 in the expression for the polar vector and dividing by the length of the vector.

As an example $(f\phi)(\phi a)$ is the polar vector of the orthogonal trajectories of $a = \text{constant}$. If we denote arc length along one of these curves by σ ,

$$dV/d\sigma = (V\phi)(\phi a)/[(fa)^2]^{\frac{1}{2}}.$$

We have used the relation $[(f\phi)(\phi a)]^2 = (fa)^2$ which is easily shown to be true.

Let
$$k = (f\phi)(\phi a)(f\psi)(\psi a),$$

and let f in the third factor be interchanged with each element of the second factor; then

$$\begin{aligned} k &= (f\phi)(\phi f)(a\psi)(\psi a) + (f\phi)(fa)(\phi\psi)(\psi a) \\ &= 2(\psi a)^2 - k. \end{aligned}$$

Thus,
$$k = [(f\phi)(\phi a)]^2 = (fa)^2.$$

9. *The principal curvature vector.* The polar vector of the curves $a = \text{constant}$ may be written either in the form (fa) or $(f\phi)(\psi\phi)(\psi a)$, and from each of these forms it is possible to determine the curvature vector of the curves $a = \text{constant}$.

If we find the curvature vector $d\mathbf{t}/ds$ from the equation

$$\mathbf{t} = (f\phi)(\psi\phi)(\psi a)/[(fa)^2]^{\frac{1}{2}},$$

$$d\mathbf{t}/ds = (\psi\phi)(\psi a)((f\phi)a)/(fa)^2 + \text{a vector in the base plane.}$$

The first term is normal to the base plane, and the others lie in the base plane.

The base component is more easily obtained from $\mathbf{t} = (fa)/[(fa)^2]^{\frac{1}{2}}$, and in this case

$$\frac{d\mathbf{t}}{ds} = \frac{((fa)a)}{(fa)^2} - \frac{(\phi a)((\phi a)a)}{[(\phi a)^2]^2} (fa).$$

Let $((fa)a)$ be resolved into its base and normal components,

$$((fa)a) = p(fa) + q(f\phi)(\phi a) + N.$$

Since (fa) and $(f\phi)(\phi a)$ are orthogonal to each other, we may multiply this equation first by (fa) and then by $(f\phi)(\phi a)$ and find that

$$p = \frac{((\psi a)a)(\psi a)}{(fa)^2}, \quad q = \frac{((\psi a)a)(\psi\phi)(\phi a)}{(fa)^2}.$$

Now,

$$\frac{dt}{ds} = \frac{((\psi a)a)(\psi\phi)(\phi a)}{[(fa)^2]^2} (f\theta)(\theta a) + N.$$

From these two expressions for the curvature vector the base component and the normal component may be obtained, and

$$(20) \quad \frac{dt}{ds} = \frac{(\psi\phi)(\phi a)((\psi a)a)}{[(fa)^2]^2} (f\theta)(\theta a) + \frac{(\psi\phi)(\psi a)((f\phi)a)}{(fa)^2}.$$

If we denote the vector of unit length in the direction of $(f\phi)(\phi a)$ by τ and the normal component of the curvature by α , (20) becomes $\Gamma_1 a \tau + \alpha$ where $\Gamma_1 a$ is the differential parameter

$$\frac{(\psi\phi)(\phi a)((\psi a)a)}{[\sqrt{(fa)^2}]^3}.$$

When $f_1 = \partial f / \partial u$, $f_2 = \partial f / \partial v$, the differential parameter $\Gamma_1 a$ is the geodesic curvature of the curves $a = \text{constant}$. Generalizing this idea we call $\Gamma_1 a$ the geodesic curvature of the curves $a = \text{constant}$ *relative to the basis* $[f_1, f_2]$.

The curves $a = \text{constant}$ determined by $\Gamma_1 a = 0$ will be called the geodesic of the surface, relative to the basis $[f_1, f_2]$. These will be called merely *geodesics* and *geodesic curvature* hereafter, the relation to the basis being understood.

10. *Other curvature vectors.* If we use σ to denote arc length along the orthogonal trajectories of the curves $a = \text{constant}$, we are able to find, by methods similar to those used above, an expression for $dt/d\sigma$; also, expressions for $d\tau/ds$ and $d\tau/d\sigma$ may be found.

If we let

$$(21) \quad \begin{aligned} \Gamma_2 a &= \frac{(f\phi)(fa)(\theta a)((\phi a)\theta)}{[\Delta_1 a]^{3/2}}, & \beta &= \frac{(\phi a)(\theta a)((f\phi)\theta)}{\Delta_1 a}, \\ \gamma &= \frac{(\psi\phi)(\psi a)(\theta a)((f\phi)\theta)}{\Delta_1 a}, & \delta &= \frac{(\phi a)((f\phi)a)}{\Delta_1 a}, \end{aligned}$$

where $\Delta_1 a = (fa)^2$, the entire list of curvature vectors may be written

$$\begin{aligned} dt/ds &= \Gamma_1 a \tau + \alpha, & dt/d\sigma &= -\Gamma_2 a \tau + \gamma, \\ d\tau/ds &= \Gamma_2 a \tau + \beta, & d\tau/d\sigma &= -\Gamma_1 a \tau + \delta. \end{aligned}$$

In these equations α and β denote the normal components of the curvature vectors of the curve $a = \text{constant}$ and its orthogonal trajectory, γ denotes the normal component of $dt/d\sigma$, and δ the normal component of $d\tau/ds$.

It is obvious that the normal vectors α , β , γ , and δ are invariant under a change of parameters but depend on the function a .

We can readily obtain relations connecting these vectors.

If in $((f\phi)\phi)(\psi a)$, the last element in the first factor be exchanged with each of the elements of the second factor,

$$((f\phi)\phi)(\psi a) = ((f\phi)\psi)(\phi a) + ((f\phi)a)(\psi\phi).$$

It is recognized immediately that

$$(23) \quad \alpha + \beta = ((f\phi)\phi)(\psi a)^2/\Delta_1 a = ((f\phi)\phi) = h^*.$$

Thus, $\alpha + \beta$ is an invariant proper and is independent of the function a . This is a well-known relation for the ordinary case.

Another relation for the ordinary case is

$$\delta - \gamma = 0.*$$

In the general representation, this equation no longer holds as we shall now see.

If the numerator of the normal component of $d\mathbf{t}/d\sigma$ be simplified,

$$\begin{aligned} (\psi\phi)(\psi a)(\theta a)((f\phi)\theta) &= (\psi\phi)(\theta a)[((f\phi)\psi)(\theta a) + ((f\phi)a)(\psi\phi)] \\ &= (\psi\phi)((f\phi)\psi)(\theta a)^2 + (\psi\phi)(\psi\theta)(\theta a)((f\phi)a) \end{aligned}$$

where the θ of the last factor has been interchanged with each element of the second factor.

In accordance with the notation used in (16),

$$(\psi\phi)(\psi a)\theta a)((f\phi)\theta) = -\mathbf{j}(\theta a)^2 + (\phi a)((f\phi)a)$$

$$\text{and } (\psi\phi)(\psi a)(\theta a)((f\phi)\theta)/\Delta_1 a = -\mathbf{j} + [(\phi a)((f\phi)a)]/\Delta_1 a.$$

Therefore, it is noted in this general case that

$$(24) \quad \delta - \gamma = \mathbf{j}.$$

Thus the vector $\delta - \gamma$ is also independent of the function a .

11. *Polar vectors to other curves.* Any vector function which is a polar vector to a curve $a = \text{constant}$ on the surface may be written in the form $p\mathbf{t} + q\boldsymbol{\tau}$, where p and q are functions of the coördinates. Let $p^2 + q^2 = 1$, then the polar vector at each point of the curve is a unit vector.

In this discussion, we shall let $p\mathbf{t} + q\boldsymbol{\tau}$ be the polar vector of the curves

* L. Ingold, *loc. cit.*, p. 588.

$b = \text{constant}$, and shall denote by s and σ the arc length along b and along the orthogonal trajectories of $b = \text{constant}$, respectively.

The curvature vector of the curves defined by this polar vector,

$$p\mathbf{t} + q\boldsymbol{\tau} \equiv p \frac{(fa)}{[(fa)^2]^{\frac{1}{2}}} + q \frac{(f\phi)(\phi a)}{[(fa)^2]^{\frac{1}{2}}},$$

has the form

$$\begin{aligned} \frac{d[p\mathbf{t} + q\boldsymbol{\tau}]}{ds} &= p \frac{(p\mathbf{t} + q\boldsymbol{\tau}, a)}{[(fa)^2]^{\frac{1}{2}}} + q \frac{(p\mathbf{t} + q\boldsymbol{\tau}, \phi)(\phi a)}{[(fa)^2]^{\frac{1}{2}}} \\ &= p \frac{d[p\mathbf{t} + q\boldsymbol{\tau}]}{ds_a} + q \frac{d[p\mathbf{t} + q\boldsymbol{\tau}]}{d\sigma_a} \end{aligned}$$

where s_a and σ_a denote arc length along $a = \text{constant}$ and the orthogonal trajectories of $a = \text{constant}$, respectively.

We may list the curvature vectors of the curves defined by the orthogonal unit vector, $q\mathbf{t} - p\boldsymbol{\tau}$, and the cross curvatures as follows:

$$\frac{d[q\mathbf{t} - p\boldsymbol{\tau}]}{d\sigma} = q \frac{d[q\mathbf{t} - p\boldsymbol{\tau}]}{ds_a} - p \frac{d[q\mathbf{t} - p\boldsymbol{\tau}]}{d\sigma_a},$$

$$\frac{d[q\mathbf{t} - p\boldsymbol{\tau}]}{ds} = p \frac{d[q\mathbf{t} - p\boldsymbol{\tau}]}{ds_a} + q \frac{d[q\mathbf{t} - p\boldsymbol{\tau}]}{d\sigma_a},$$

$$\frac{d[p\mathbf{t} + q\boldsymbol{\tau}]}{d\sigma} = q \frac{d[p\mathbf{t} + q\boldsymbol{\tau}]}{ds_a} - p \frac{d[p\mathbf{t} + q\boldsymbol{\tau}]}{d\sigma_a}.$$

When these equations are expanded, the curvature vectors become

$$\frac{d[p\mathbf{t} + q\boldsymbol{\tau}]}{ds} = G_1(q\mathbf{t} - p\boldsymbol{\tau}) + p^2\boldsymbol{\alpha} + pq(\boldsymbol{\gamma} + \boldsymbol{\delta}) + q^2\boldsymbol{\beta},$$

$$\frac{d[q\mathbf{t} - p\boldsymbol{\tau}]}{d\sigma} = G_2(p\mathbf{t} + q\boldsymbol{\tau}) + q^2\boldsymbol{\alpha} - (\boldsymbol{\gamma} + \boldsymbol{\delta}) + p^2\boldsymbol{\beta},$$

(25)

$$\frac{d[q\mathbf{t} - p\boldsymbol{\tau}]}{ds} = -G_1(p\mathbf{t} + q\boldsymbol{\tau}) + pq\boldsymbol{\alpha} + q^2\boldsymbol{\gamma} - p^2\boldsymbol{\delta} - pq\boldsymbol{\beta},$$

$$\frac{d[p\mathbf{t} + q\boldsymbol{\tau}]}{d\sigma} = -G_2(q\mathbf{t} - p\boldsymbol{\tau}) + pq\boldsymbol{\alpha} + q^2\boldsymbol{\delta} - p^2\boldsymbol{\gamma} - pq\boldsymbol{\beta},$$

where

$$G_1 = -\frac{(qa)}{[\Delta_1 a]^{\frac{1}{2}}} + \frac{(p\phi)(\phi a)}{[\Delta_1 a]^{\frac{1}{2}}} - p\Gamma_1 + q\Gamma_2,$$

and

$$G_2 = -\frac{(pa)}{[\Delta_1 a]^{\frac{1}{2}}} - \frac{(q\phi)(\phi a)}{[\Delta_1 a]^{\frac{1}{2}}} + q\Gamma_1 + p\Gamma_2.$$

If $p = \cos \theta$ and $q = \sin \theta$, the normal components of (25) in the new system are:

$$\begin{aligned}
 \bar{\alpha} &= \alpha \cos^2 \theta + (\gamma + \delta) \sin \theta \cos \theta + \beta \sin^2 \theta, \\
 \bar{\beta} &= \alpha \sin^2 \theta - (\gamma + \delta) \sin \theta \cos \theta + \beta \cos^2 \theta, \\
 \bar{\gamma} &= (\alpha - \beta) \sin \theta \cos \theta + \gamma \cos^2 \theta - \delta \sin^2 \theta, \\
 \bar{\delta} &= (\alpha - \beta) \sin \theta \cos \theta + \delta \cos^2 \theta - \gamma \sin^2 \theta.
 \end{aligned}
 \tag{26}$$

From this we get

$$\bar{\alpha} + \bar{\beta} = \alpha + \beta = h^*,$$

$\alpha + \beta$ is therefore independent of the angle θ .

The vector $\bar{\delta} - \bar{\gamma}$ is also invariant, that is

$$\bar{\delta} - \bar{\gamma} = \delta - \gamma = j.$$

In (25) we denoted by G_1 the geodesic curvature of the curves having $pt + q\tau$ as the polar vector and by G_2 the geodesic curvature of the curves having $qt - p\tau$ as the polar vector, i. e.,

$$\begin{aligned}
 G_1 &= - (dq/ds_a) + (dp/d\sigma_a) - p\Gamma_1 + q\Gamma_2, \\
 G_2 &= - (dp/ds_a) - (dq/d\sigma_a) + q\Gamma_1 + p\Gamma_2.
 \end{aligned}
 \tag{27}$$

If $p = \cos \theta$ and $q = \sin \theta$,

$$\begin{aligned}
 (dp/ds_a) &= -\sin \theta (d\theta/ds_a) = -q (d\theta/ds_a), \\
 (dq/d\sigma_a) &= \cos \theta (d\theta/d\sigma_a) = p (d\theta/d\sigma_a), \\
 (dq/ds_a) &= \cos \theta (d\theta/ds_a) = p (d\theta/ds_a), \\
 (dp/d\sigma_a) &= -\sin \theta (d\theta/d\sigma_a) = -q (d\theta/d\sigma_a).
 \end{aligned}$$

We know that

$$(dV/ds) = p(dV/ds_a) + q(dV/d\sigma_a)$$

$$\text{and} \quad (dV/d\sigma) = q(dV/ds_a) - p(dV/d\sigma_a).$$

Therefore, by substituting these results in the equations for G_1 and G_2 ,

$$\begin{aligned}
 G_1 &= - (d\theta/ds) - p\Gamma_1 + q\Gamma_2, \\
 G_2 &= (d\theta/d\sigma) + q\Gamma_1 + p\Gamma_2.
 \end{aligned}
 \tag{28}$$

This is a generalization of a known result.*

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* See Eisenhart, *Differential Geometry*, p. 150, Ex. 17.

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Vita

Nola Lee Anderson was born January 9, 1897, near Bucklin, Missouri, and her secondary education was received in that town. She entered the University of Missouri in the fall of 1919 and received the degree of B. S. in Education in 1922.

During the years 1922-24 she taught mathematics in the high school at St. Charles, Missouri, and the following year had charge of the department of mathematics in Central College for Women, Lexington, Missouri.

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